

# Random spin distributions and the diffusion equation

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We show that the probability distribution corresponding to a fully random tracial state of a system of spin- $S$  particles satisfies a diffusion-like equation. The diffusion coefficient turns out to be equal to  $S(S+1)/6$ , where  $S$  is the magnitude of the spin of each particle.

Many-spin systems exhibit interesting properties [1], which explains the reason for which they are among the most studied systems in current research. These studies are motivated by several potential applications in mesoscopic and atomic physics, among which we mention the promising quantum technologies that are slowly emerging. The theoretical investigation of the dynamics of spin systems requires, very often, the use of statistical and probabilistic techniques. In particular, the probability distribution associated with the addition of the spin degrees of freedom turns out to be of great importance and usefulness. For example, we have employed random distributions of spin angular momentum to deal with the description of the decoherence and the entanglement evolution of qubits interacting with spin environments. In this paper we address the relation between these Gaussian distributions and the diffusion equation [2, 3].

To be more explicit in our discussion, we denote by  $\vec{J}$  the sum of the individual spins of a set of  $N$  spin- $S$  particles, i.e.,  $\vec{J} = \sum_{i=1}^N \vec{S}_i$ , and by  $\lambda_j = j(j+1)$  and  $m$  the eigenvalues of the operators  $J^2$  and  $J_z$  respectively. This means that, given a complete set  $\{|j, m\rangle\}$  of common eigenvectors of the above operators, we may write  $J^2|j, m\rangle = j(j+1)|j, m\rangle$ , and  $J_z|j, m\rangle = m|j, m\rangle$ . For a given  $j$ , the multiplicity of  $m$  is simple to calculate

and is equal to  $2j+1$ . On the other hand, the degeneracy  $\nu(N, j; S)$  corresponding to the quantum number  $j$  is shown to satisfy the relation:

$$\nu(j, N+1; S) = \sum_{j'=|j-S|}^{j'+S} \nu(j', N; S). \quad (1)$$

The latter equation will be our starting point to show that the probability distribution verifies a diffusion-like equation when the number of spins is sufficiently large. To begin we note that it is more convenient to deal with each of the components of the vector operator  $\vec{J}$ ; the reasoning could equally be carried out in terms of vector quantities but the notation will be a little cumbersome. This being said, let  $P(m, N)$  be the probability associated with the eigenvalue  $m$  of the operator  $J_z$ . By noting that the multiplicity of  $S_z$  is  $2S+1$ , we infer from Eq.(1) that the following equality holds:

$$(2S+1)P(m, N+1) = \sum_{\rho=-S}^S P(m-\rho, N). \quad (2)$$

For large  $N$  and  $m$ , we may expand in Taylor series, up to second order, both  $P(m, N+1)$  and  $P(m-\rho, N)$  to obtain

$$(2S+1) \left[ P(m, N) + \frac{\partial P}{\partial N} + \frac{1}{2} \frac{\partial^2 P}{\partial N^2} \right] = \sum_{\rho=-S}^S P(m, N) - \frac{\partial P}{\partial m} \sum_{\rho=-S}^S \rho + \frac{1}{2} \frac{\partial^2 P}{\partial m^2} \sum_{\rho=-S}^S \rho^2 \quad (3)$$

On account of the fact that

$$\sum_{\rho=-S}^S \rho^2 = \frac{1}{3} S(2S+1)(S+1), \quad (4)$$

we end up with the equation

$$\frac{\partial P}{\partial N} + \frac{1}{2} \frac{\partial^2 P}{\partial N^2} = \frac{1}{6} S(S+1) \frac{\partial^2 P}{\partial m^2}. \quad (5)$$

It is physically justified to neglect the second term in the

left-hand side of the above equation because

$$\left| \frac{\partial P}{\partial N} \right| \gg \left| \frac{\partial^2 P}{\partial N^2} \right|$$

when  $N$  is large. Whence:

$$\frac{\partial P}{\partial N} = \frac{1}{6} S(S+1) \frac{\partial^2 P}{\partial m^2}, \quad (6)$$

which reminds us the diffusion equation in one dimension without drift, namely, [4]

$$\frac{\partial \rho(x, t)}{\partial t} = D \frac{\partial^2 \rho(x, t)}{\partial x^2}, \quad (7)$$

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where  $D$  is the diffusion equation. Thus it is tempting to interpret Eq.(6) as being a diffusion equation where the number of spins plays the role of time, and the role of the position  $x$  is played by the magnitude of the spin  $m$ . The corresponding diffusion coefficient is

$$D \equiv \frac{1}{6}S(S+1). \quad (8)$$

The solution of Eq.(6) should be subject to the boundary condition

$$P(m, 0) = 0, \quad (9)$$

which is physically obvious. The corresponding explicit form can be found by noting that the solution of Eq.(7) is given by

$$\rho(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/(4Dt)}. \quad (10)$$

Hence, by comparison, we obtain:

$$P(m, N) = \sqrt{\frac{3}{2\pi NS(S+1)}} e^{-\frac{3m^2}{2NS(S+1)}}, \quad (11)$$

which is the desired result.

We have already shown how to derive the probability distribution of the random variables associated with the spin operators  $J_x/\sqrt{N}$ ,  $J_y/\sqrt{N}$ , and  $J_z/\sqrt{N}$  (which are the components of the operator  $\vec{J} = \sum_{i=1}^N \vec{S}_i$ ) when  $N \rightarrow \infty$  using the trace properties of angular momentum for  $S = 1/2$  [2]. Again, the quantity  $N$  refers to the total number of the spin- $\frac{1}{2}$  particles, meaning that for finite  $N$ , the dimension of the total spin space is  $2^N$ . For an arbitrary value of the spin  $S$ , the same method may be employed to obtain the above results as follows:

The trace of even powers of the components of the total spin operator is given by

$$\begin{aligned} \text{tr}(\hat{J}_z)^{2\ell} &= \frac{2\ell!}{\underbrace{2! \times 2! \cdots 2!}_{\ell \text{ terms}}} \sum_{\Pi_\ell} \left( \prod_{k=1}^{\ell} \text{tr} S_{zk}^2 \prod_{k=\ell+1}^N \text{tr} \mathbb{I}_{2S+1} \right)_{\Pi_\ell[1,2,\dots,N]} \\ &+ Q_{\ell-1}(N), \end{aligned} \quad (12)$$

where  $\mathbb{I}_{2S+1}$  refers to the  $2S+1$  dimensional unit matrix, and the products should be evaluated for all the possible partitions  $\Pi_\ell[1, 2, \dots, N]$  of  $N$  elements into subsets of  $\ell$  elements; the quantity  $Q_{\ell-1}(N)$  is a polynomial in  $N$  whose degree is at most equal to  $\ell-1$ . The latter equation means that

$$\text{tr}(\hat{J}_z)^{2\ell} = N^\ell \left[ \frac{(2\ell)!}{2^\ell} (2S+1)^{N-\ell} \left[ \frac{1}{3} S(S+1)(2S+1) \right]^\ell + O\left(\frac{1}{N}\right) \right]. \quad (13)$$

By rescaling the spin operator and taking the limit  $N \rightarrow$

$\infty$ , we obtain

$$\lim_{N \rightarrow \infty} (2S+1)^{-N} \text{tr}(\hat{S}_z/\sqrt{N})^{2\ell} = \frac{(2\ell)!}{2^\ell} \left[ \frac{1}{3} S(S+1) \right]^\ell. \quad (14)$$

The characteristic function, with moments given by the right-hand side of Eq.(14) is

$$\begin{aligned} \Phi(t) &= \sum_{n=0}^{\infty} (it)^{2n} \frac{(2n)!}{2^{3n} n! (2n)!} = \sum_{n=0}^{\infty} \frac{(it)^{2n}}{n!} \left[ \frac{1}{6} S(S+1) \right]^n \\ &= \exp\left\{ -\frac{1}{6} S(S+1) t^2 \right\}. \end{aligned} \quad (15)$$

By Fourier transforming the characteristic function

$$P(m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(t) e^{-itm} dt \quad (16)$$

and making the substitution  $m \rightarrow m/\sqrt{N}$  we obtain exactly Eq.(11).

The joint distribution of three such identical independent random variables  $m_1, m_2, m_3$  will be

$$\begin{aligned} P(m_1, m_2, m_3) &= p(m_1)p(m_2)p(m_3) \\ &= \left[ \frac{3}{2\pi NS(S+1)} \right]^{3/2} e^{-\frac{3(m_1^2 + m_2^2 + m_3^2)}{2NS(S+1)}} \end{aligned} \quad (17)$$

Furthermore if we require that  $\sqrt{m_1^2 + m_2^2 + m_3^2} = j$ , then the probability distribution becomes [3]

$$\tilde{P}(m_1, m_2, m_3) \equiv P(j) = 4\pi j^2 P(m_1, m_2, m_3). \quad (18)$$

To summarize, we have shown that the probability distribution corresponding to a random tracial state of a set of  $N$  spin- $S$  particles satisfies a diffusion-like equation when  $N$  is large. We have used this fact to derive

its explicit form, and extended the use of the method developed in Re. [2] to arbitrary values of the spin.

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